DISCRETE SPECTRUM OF QUANTUM TUBES

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A quantum tube is essentially a tubular neighborhood about an immersed complete manifold in some Euclidean space. To be more precise, let $\Sigma \hookrightarrow \mathbb{R}^{n+k}$, $k \geq 1$, $n = dim(\Sigma)$, be an isometric immersion, where Σ is a complete, noncompact, orientable manifold. Then consider the resulting normal bundle $T^{\perp}\Sigma$ over Σ, and the submanifold $F = \{(x,\xi)|x \in \Sigma, |\xi < r\} \subset T^{\perp}\Sigma$ for r small enough. The quantum tube is defined as the Riemannian manifold $(F, f^*(ds_E^2))$, where ds_E^2 is the Euclidean metric in $\mathbb{R}^{n+k}$ and the map f is defined by $f(x,\xi) = x + \xi$. If $k = 1$, then the quantum tube is also called the quantum layer. The immersion of Σ means that the resulting image of F under f in $\mathbb{R}^{n+k}$ can have intersections. Moreover, since Σ can have quite complicated topology in general, $f(F)$ can too. However, by doing our analysis on F directly (with the pull-back metric), these complications are naturally bypassed (cf. [1, 5]).

Although on noncompact, noncomplete manifolds there is no unique self-adjoint extension of the Laplacian acting on compactly supported functions, we can always, via the Dirichlet quadratic form define the *Dirchlet Laplacian* Δ_D, which is the self-adjoint extension that reduces to the self-adjoint Laplacians defined on complete manifolds and compact manifolds with Dirichlet boundary conditions. Therefore we can proceed to perform spectral analysis, in particular, on the quantum tube. Geometers, like physicists, are first and foremost interested in the existence and distribution of the discrete spectrum. For noncompact manifolds this is in general not an easy task at all. However, using standard variational techniques, the authors Duclos, Exner, and Krejčiřík were able to, in an interesting paper [2], prove the existence of discrete spectra for the quantum layer (corresponding to $n = 2$ and

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$k = 1$ in our defintion) under certain integral-curvature conditions on Σ. Since the discrete spectrum are isolated eigenvalues of finite multiplicity, their result is even better, especially in the physical sense since the discrete spectrum is composed of energy levels of bound states of a nonrelativistic particle. Our definition of the quantum tube improved theirs in [2] and we were able to generalize the same existence result to the quantum tube. The challenges in our attempt at generalization were mainly geometrical, as we sought to replace the necessary geometric conditions with appropriate higher dimensional analogs so that similar variational techniques from [2] can be applied meaningfully. One notable observation that arised is the sharp contrast between parabolic and non-parabolic manifolds.

The main result in [5] is as follows:

Theorem 1. *Let $n \geq 2$ be a natural number. Suppose $\Sigma \subset R^{n+1}$ is a complete immersed parabolic hypersurface such that the second fundamental form $A \to 0$ at infinity. Moreover, we assume that*

$$\sum_{k=1}^{[n/2]} \mu_{2k} \mathrm{Tr}(\mathcal{R}^k) \text{ is integrable and } \quad \int_\Sigma \sum_{k=1}^{[n/2]} \mu_{2k} \mathrm{Tr}(\mathcal{R}^k) d\Sigma \leq 0, \tag{1}$$

where $\mu_{2k} > 0$ for $k \geq 1$ are positive computable coefficients; $[n/2]$ is the integer part of $n/2$, and $\mathcal{R}^k$ is the induced endomorphism of $\Lambda^{2k}(T_x\Sigma)$ by the curvature tensor $\mathcal{R}$ of Σ. Let a be a positive real number such that $a||A|| < C_0 < 1$ for a constant C_0. If Σ is not totally geodesic, then the ground state of the quantum layer Ω exists.

In [6], we generalized the above results to high codimensional cases:

Theorem 2. *Let $(F, f^*(ds_E^2))$ be an order-k quantum tube with radius r and base manifold Σ of dimension n such that the second fundamental form goes to zero at infinity. Moreover, we assume that Σ is a parabolic manifold, $\sum_{p=1}^{[n/2]} \mu_{2p} \mathrm{Tr}(\mathcal{R}^k)$ integrable, and*

$$\int_\Sigma \sum_{p=1}^{[n/2]} \mu_{2p} \mathrm{Tr}(\mathcal{R}^k) d\Sigma \; \leq \; 0, \tag{2}$$

If Σ is not totally geodesic, then the ground state of the quantum tube from Σ exists.

By applying the above result into two dimensional case, we get

Corollary 1. *Suppose that Σ is a complete immersed surface of R^{n+1} such that the second fundamental form $A \to 0$. Suppose that the Gauss curvature is integrable and suppose that*

$$e(\Sigma) - \sum \lambda_i \leq 0, \tag{3}$$

where $e(\Sigma)$ is the Euler characteristic number of Σ; λ_i is the isoperimetric constant at each end of Σ, defined as

$$\lambda_i = \lim_{r\to\infty} \frac{\mathrm{vol}(B(r))}{\pi r^2}$$

at each end E_i. Let a be a positive number such that $a||A|| < C_0 < 1$. If Σ is not totally geodesic, then the ground state of the quantum layer Ω exists. In particular, if $e(\Sigma) \leq 0$, then the ground state exists.

We remark here that in the proof of Theorem 2 (and so as in Theorem 1 and the analogous result in [2]), the asymptotically flat condition on Σ ensures that we get a lower bound on the bottom of the essential spectrum, while condition 2 (along with parabolicity) enabled us to show that such a bound is also a strict upper bound for the total spectrum. In this way, we were able to conclude that the discrete spectrum must be non-empty. It seems intuitive that the asymptotically flat condition on Σ is essential for there to be discrete spectra, since only the "relatively-curved part of Σ" located in the "interior" of Σ will trap a particle. If Σ is curved more-or-less the same everywhere, then a particle may be equally likely to be anywhere since the "terrain" is more-or-less indistinguishable everywhere. The preceding is of course a physical intuition coming from the interpretation of our problem as a problem in non-relativistic quantum mechanics, however, it serves to motivate the idea that other global curvature assumptions similar to (2) may also provide the existence of ground state on quantum tubes.

From Corollary 1 (and the result in [2]), it is natural to make the following

Conjecture. *Suppose Σ is an embedded asymptotically flat surface in R^3 which is not totally geodesic and the Gauss curvature is integrable. Then the ground state of the quantum layer built from Σ exists.*

We have partial results in this direction [8]:

Theorem 3 (Lu)**.** *Suppose Σ is asymptotically flat but not totally geodesic in R^3. If the Gauss curvature of Σ is positive, then the ground state exists for the quantum layer.*

In general, we have the following result:

Theorem 4 (Lu)**.** *Suppose Σ is asymptotically flat but not totally geodesic in R^3 and suppose the Gauss curvature is integrable. Let H be the mean curvature. If there is an $\varepsilon > 0$ such that*

$$\overline{\lim_{r\to\infty}}\, \frac{1}{r}\left|\int_{B(r)} H d\Sigma\right| \geq \varepsilon, \tag{4}$$

then the ground state of the quantum layer exists.

Let's make some remarks on the above results. By the work of [2], we only need to prove the conjecture under the assumption that

$$\int_\Sigma K d\Sigma > 0.$$

By a result of Hartman [4], we know that

$$\frac{1}{2\pi}\int_\Sigma K d\Sigma = e(\Sigma) - \sum \lambda_i. \tag{5}$$

Thus we have $e(\Sigma) > 0$, or $e(\Sigma) \geq 1$. Let $g(\Sigma)$ be the genus of Σ, we then know $g(\Sigma) = 0$ and Σ must be differmorphic to $\mathbb{R}^2$, which is a very strong topological restriction.

On the other hand, we have the following lemma:

Lemma 1. *Under the assumption that $\int_\Sigma K d\Sigma > 0$, there is an $\varepsilon > 0$ such that*

$$\overline{\lim_{r\to\infty}}\, \frac{1}{r}\int_{B(r)} |H| d\Sigma \geq \varepsilon.$$

Proof. Since Σ is differmorphic to $\mathbb{R}^2$, by (5)

$$0 < \int_\Sigma K d\Sigma \leq 2\pi < 4\pi.$$

Thus by a theorem of White [9], we get the conclusion.

□

We believe (4) is true under the same assumption as in the Lemma.

The above results confirmed the belief that the spectrum of the quantum tube only depends on the geometry of Σ, its base manifold. With regard to the geometry of Σ (or any complete, noncompact manifold for that matter), the volume growth (of geodesic balls) is an important

geometric property. Roughly speaking, complete, noncompact manifolds can be separated into those with at most quadratic volume growth and those with faster volume growth. They are termed (very roughly) parabolic and non-parabolic, respectively. It is the property of parabolicity assumed on Σ that allowed us to prove the existence of discrete spectra on quantum tubes. However, if one looks at the hypothesis of Theorem 2, where Σ is required to have vanishing curvature at infinity while being immersed in Euclidean space, it is highly likely that Σ will not be of at most quadratic volume growth if $\dim(\Sigma) > 2$, hence unlikely to be parabolic. However, one can be sure that the set of base manifolds satisfying the hypothesis of Theorem 2 is not empty, due to an example provided in [5]. Nevertheless, it is clear that if one were to maintain the assumption of asymptotic flatness of Σ, then one should begin paying attention to the situation when Σ is non-parabolic.

Although we do not yet have a result specifically for quantum tubes over non-parabolic manifolds, there is the following preliminary result for general (possibly non-parabolic) base manifolds (see [7]):

Theorem 5. *Suppose Σ is not totally geodesic, satisfies the volume growth $V(r) \leq Cr^m$, and whose second fundamental form $\vec{A}$ goes to zero at infinity and decays like $r^2\|\vec{A}\| \to 0$ as $r \to \infty$. Moreover, suppose*

$$\lim_{R\to\infty} \frac{1}{R^{m-2}} \int_{B(R)} \sum_{p=1}^{[n/2]} \mu_{2p} K_{2p} \tag{6}$$

exists(possibly $-\infty$) and strictly less than $-\frac{1}{4}CC_1m^2e^2$, where C_1 is an explicit constant that depends on the dimension of Σ, radius of the quantum tube, and the upper bound on the curvature of Σ. Then the discrete spectrum of the quantum tube with base manifold Σ is non-empty.

The result above is certainly an overkill if Σ is parabolic. Thus we should think of applying it only to the case of non-parabolic Σ, where $m > 2$. The direct application of the volume growth hypothesis allows one to use polynomially decaying test functions to obtain the condition on (6), and in turn obtain the upper-bound for the bottom of the total spectrum.

Theorem 5 is only a first step towards generalizing the phenomenon of localization (we mean this to be the existence of ground state) to quantum tubes over non-parabolic manifolds with similar non-positivy

assumptions on curvature as the parabolic case. One clearly cites the technical assumption on the decay rate of the second fundamental form, and one would like to remove it. In addition, the negativity condition on (6) is very strong. We do not yet know if weaker assumptions such as (2) are applicable to the case where Σ is non-parabolic.

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